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On SSQM and the U(N) non-linear Schrödinger equation

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Abstract. The method for obtaining the superpartner potential in supersymmetric quantum mechanics (SSQM) is discussed in connection with non-linear equations and reflectionless potentials. The correspondence between a new class of soliton solutions to the U(N) non-linear Schrödinger equation, obtained via application of SSQM, and previously known soliton solutions is also discussed.

1. Introduction

The study of the particle-like behaviour of non-linear fields, originally developed by Einstein to systematically derive the equations of motion for a particle in an external field, took a new turn with the discovery of soliton solutions (Dodd *et al* 1982). Soliton-type properties have since been found in a great variety of non-linear physical systems such as Korteveg-de Vries (KdV), sine-Gordon, non-linear Schrödinger (NLS), etc.

In the 1970s theoretical physics developed a new fruitful concept in supersymmetry, i.e. the concept of treating bosons and fermions equally (Bagger and Wess 1983). The interesting advantage of supersymmetry is that it provides a natural way of incorporating fermions into the soliton system; it was first done for non-linear equations via direct supersymmetrisation by Di Vecchia and Ferrara (1977) and Hruby (1977).

From this supersoliton theory, which is given by the supersoliton Lagrangian in $(1+1)$ spacetime dimensions:

$$L=\tfrac{1}{2}[(\partial_\mu\phi)^2-V^2(\phi)+\bar{\psi}(\mathrm{i}\not{\partial}+V'(\phi))\psi] \tag{1.1}$$

where ϕ is a Bose field and ψ is a Fermi field, we can obtain SSQM by restricting to $(0+1)$ spacetime dimension (the prime denotes differentiation with respect to the argument).

If we substitute into (1.1) the following restriction:

$$\phi\to x(t) \qquad \partial_\mu\to\partial_t$$
$$\bar{\psi}\to\psi^{\mathrm{T}}\sigma_2 \qquad \mathrm{i}\not{\partial}\to\mathrm{i}\partial_t\sigma_2$$

where

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

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with components being interpreted as anticommuting c-numbers, σ_k denote the Pauli matrices, then $L \to L_{\mathrm{SSQM}}$ and

$$L_{\mathrm{SSQM}} = \tfrac{1}{2}[(\partial_t x)^2 - V^2(x) + \psi^{\mathrm{T}}(\mathrm{i}\partial_t + \sigma_2 V'(x))\psi]. \tag{1.2}$$

The corresponding Hamiltonian has the known form

$$H_{\mathrm{SSQM}} = \tfrac{1}{2}p^2 + \tfrac{1}{2}V^2(x) + \tfrac{1}{2}\mathrm{i}[\psi_1, \psi_2]V'(x) \tag{1.3}$$

which was proposed by Witten (1981) and also by Salamonson and Van Holten (1982).

In the theory of supersolitons the method of inverse scattering plays a crucial role. It is interesting to show the role of the methods known from the solitons and non-linear wave equations in SSQM. In the present work we want to show the role of SSQM for non-linear equations such as NLS and KdV.

The application of SSQM to the Zakharov equations (Zakharov 1972) and the generalisation, first given in Makhankov (1974), are discussed. We also demonstrate the correspondence between a new class of soliton solutions for the $\mathrm{U}(N)$ NLS (Makhankov and Myrzakulov 1986) and the corresponding results in SSQM.

2. Supersymmetric quantum mechanics

We shall start with the Schrödinger factorisation in quantum mechanics (Kwong and Rosner 1986). Consider the one-dimensional Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x)\right)\psi(x) = E\psi(x) \tag{2.1}$$

and its factorisation in the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + v\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} + v\right)\psi = E\psi. \tag{2.2}$$

If we denote

$$A^{\pm} = \pm\frac{\mathrm{d}}{\mathrm{d}x} + v \tag{2.3}$$

we can write $A^+A^-\psi = E\psi = H_+\psi$, but it gives

$$A^+A^- = H_+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v^2 + v_x = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+.$$

Let us choose the ground state ψ_0^+ to satisfy $H_+\psi_0^+ = A^+A^-\psi_0^+ = 0$, implying

$$A^-\psi_0^+ = 0. \tag{2.4}$$

This is a first-order differential equation

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + v(x)\right)\psi_0^+ = 0$$

leading to

$$v = \psi_{0x}^+/\psi_0^+. \tag{2.5}$$

If we consider the factorisation in the form

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+v\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}+v\right)\psi=E\psi$$

we get

$$A^-A^+=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+v^2-v_x=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V_-=H_-. \tag{2.6}$$

Now suppose ψ^+ to be any eigenfunction of H_+

$$H_+\psi^+=E_+\psi^+ \tag{2.7}$$

then

$$A^-H_+\psi^+=E_+(A^-\psi^+)=A^-A^+A^-\psi^+.$$

Either $A^-\psi^+=0$ (so that $E_+=0$ and ψ_+ is the ground state) or $H_-(A^-\psi^+)=E_+(A^-\psi^+)$.

Thus, every eigenstate of H_+, except for the ground state, gives rise (via A^-) to an eigenstate of H_- with the same eigenvalue. The ground state of H_+ with the zero energy does not correspond to any eigenstate of H_-. This means that the Hamiltonian H_+ has the same spectrum as H_- with the addition of an extra ground state.

If we denote by ψ_0^- the solution of the zero-energy Schrödinger equation with H_-:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V_-\right)\psi_0^-=-\psi_{0xx}^-+(v^2-v_x)\psi_0^-=0 \tag{2.8}$$

we get

$$v=-\psi_{0x}^-/\psi_0^- \tag{2.9}$$

and by comparing with (2.5), we have

$$\psi_0^+\sim 1/\psi_0^-.$$

The factorisation presented here can be written in a supersymmetric way. In the matrix formulation H_{SSQM} (1.3) becomes a 2×2 matrix as well

$$H_{\mathrm{SSQM}}=\frac{1}{2}\begin{pmatrix}-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+v^2(x)+v_x(x) & 0\\ 0 & -\frac{\mathrm{d}^2}{\mathrm{d}x^2}+v^2(x)-v_x(x)\end{pmatrix}. \tag{2.11}$$

Then

$$H_{\mathrm{S}}=2H_{\mathrm{SSQM}}=\begin{pmatrix}H_+ & 0\\ 0 & H_-\end{pmatrix}=\begin{pmatrix}A^+A^- & 0\\ 0 & A^-A^+\end{pmatrix}=\{Q^-,Q^+\} \tag{2.12}$$

where the 'supercharges' are defined as

$$Q^-=\begin{pmatrix}0 & 0\\ A^- & 0\end{pmatrix}\qquad Q^+=\begin{pmatrix}0 & A^+\\ 0 & 0\end{pmatrix}.$$

The other relations are

$$(Q^-)^2=(Q^+)^2=0\qquad [H_{\mathrm{S}},Q^-]=[H_{\mathrm{S}},Q^+]=0. \tag{2.13}$$

The eigenfunctions of H_S are

$$\psi_S = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$$

and they have the properties

$$Q^- \psi_S = \begin{pmatrix} 0 \\ \psi^- \end{pmatrix} \qquad \text{unless } A^- \psi^+ = 0$$

$$Q^+ \psi_S = \begin{pmatrix} \psi^+ \\ 0 \end{pmatrix}.$$

We can call the levels $\binom{\psi^+}{0}$ 'bosonic' and the levels $\binom{0}{\psi^-}$ 'fermionic' in the view of the 'fermionic' nature of the 'superalgebra' in the relations (2.12) and (2.13).

In the theory of the spectral transforms and solitons (Calogero and Degasperis 1982) it is shown that the Schrödinger factorisation (2.2) is equivalent to the Muira transformation between V_+ and v:

$$V_+ = v^2 + v_x$$

coupling KdV and modified KdV (MKdV). The same is valid for V_- because MKdV is invariant under the transformation $v \to -v$. In this sense the Miura transformation represents the supersymmetric 'square root'.

There exists a deep connection between SSQM and the N-soliton solution of the KdV, reflectionless potentials

$$U_N(x) = -N(N+1)b^2 \operatorname{sech}^2 bx \qquad N = 1, 2, \ldots . \tag{2.14}$$

(i) Let us take $N = 1$, $b = 1/L\sqrt{2}$ in the symmetric reflectionless potential (2.14)

$$u(x) = -\frac{1}{L^2} \operatorname{sech}^2 \frac{x}{L\sqrt{2}}.$$

Then, $u(x)$ can be regarded as a one-soliton solution of the KdV equation for $t = 0$, i.e. of the equation

$$u_t - 6vu_x + u_{xxx} = 0. \tag{2.15}$$

The KdV one-soliton solution for all t is

$$u(x, t) = -\frac{1}{L^2} \operatorname{sech}^2 \left(\frac{x - (2/L^2)t}{L\sqrt{2}} \right).$$

The same is valid for higher N.

(ii) Let us consider now a function $v(x, t)$ satisfying MKdV

$$v_t + 6\left(\frac{1}{2L^2} - v^2 \right) v_x + v_{xxx} = 0. \tag{2.16}$$

Then, if we define

$$V_- = v^2 - v_x - 1/2L^2 \tag{2.17}$$

as is usual in SSQM, it can easily be shown that V_- satisfies KdV. The same is valid for

$$V_+ = v^2 + v_x - 1/2L^2. \tag{2.18}$$

In this way we can see that there exists a general connection between N-soliton solutions of KdV, SSQM, the inverse scattering method and the construction of the reflectionless potentials.

We shall now be concerned with the application of these results of SSQM to the Schrödinger equation with self-consistent potentials.

3. The application of SSQM to non-integrable systems

Here we discuss the non-integrable system (Zakharov 1972)

$$\mathrm{i}\psi_t + \psi_{xx} - \eta\psi = 0 \tag{3.1a}$$

$$\eta_{tt} - \eta_{xx} = |\psi|^2_{xx} \tag{3.1b}$$

and the system (Makhankov 1974)

$$\mathrm{i}\psi_t + \psi_{xx} - \eta\psi = 0 \tag{3.2a}$$

$$\eta_{tt} - \eta_{xx} - \alpha(\eta^2)_{xx} - \beta\eta_{xxxx} = |\psi|^2_{xx}. \tag{3.2b}$$

Here $\psi(x, t)$ and $\eta(x, t)$ are respectively complex and real functions; α and β are real parameters.

These (in general) non-integrable systems have applications in interesting areas in physics (Makhanhov *et al* 1986).

We shall now demonstrate, using (3.1), that the basic role played by (3.1a) in finding soliton-like solutions is that of being the symmetric reflectionless potential. The same will be true for system (3.2).

Now we shall discuss the so-called quasistatic limit of the Zakharov (Z) equations (3.1a) and (3.1b). Neglecting the term η_{tt}, (3.1b) has the form

$$(\eta + |\psi|^2)_{xx} = 0$$

which implies $\eta = -|\psi|^2$ if η and $|\psi|^2$ are square integrable. Substitution of this expression for η into (3.1a) yields the NLS equation

$$\mathrm{i}\psi_t + \psi_{xx} + |\psi|^2\psi = 0. \tag{3.3}$$

It is well known that (3.1a) and (3.1b) have a one-soliton solution:

$$\psi = \frac{1}{L}\operatorname{sech}\left(\frac{x - x_0 - vt}{L\sqrt{2(1-v^2)}}\right)\exp\left[\tfrac{1}{2}\mathrm{i}vx - \mathrm{i}\left(\tfrac{1}{4}v^2 - \frac{1}{2L^2(1-v^2)}\right)t + \mathrm{i}\theta_0\right] \tag{3.4a}$$

and

$$\eta = -|\psi|^2/(1 - v^2) \tag{3.4b}$$

where $L > 0$, v, x_0 and θ_0 are constants.

It is clear that solution (3.4a) tends to the particular one-soliton solution

$$\psi(x, t) = \exp\left(\frac{\mathrm{i}t}{2L^2}\right)\frac{1}{L}\operatorname{sech}\frac{x}{L\sqrt{2}} \tag{3.5}$$

and

$$\eta(x) = -|\psi|^2 \tag{3.6}$$

for $v = 0$, $x_0 = 0$ and $\theta_0 = 0$.

If we put solutions (3.5) and (3.6) into (3.1a), we obtain

$$\left(-\frac{d^2}{dx^2}-\frac{1}{L^2\cosh^2(x/L\sqrt{2})}\right)\psi_1(E_1)=E_1\psi_1(E_1) \tag{3.7}$$

i.e. the eigenvalue equation $H_1\psi_1=E_1\psi_1$ corresponding to (3.1a). In (3.7) we denote the eigenvalue by $E_1=-\gamma_1^2=-1/2L^2$ and it corresponds to the eigenfunction

$$\psi_1=\frac{1}{L}\operatorname{sech}\frac{x}{L\sqrt{2}}.$$

Now we shall consider $\eta(x)$

$$\eta(x)=-|\psi_1|^2=-\frac{1}{L^2}\operatorname{sech}^2\frac{x}{L\sqrt{2}}$$

as the symmetric reflectionless potential in the eigenvalue problem (3.7), and using the results of SSQM we shall construct other symmetric reflectionless potentials as 'superpartners'. We can see that $H_1=-d^2/dx^2+\eta_1$ is the superpartner to $H_0=-d^2/dx^2+\eta_0$, where the potential η_1 supports a simple bound state at energy $E=-1/2L^2$ while η_0 supports no bound states.

Choosing $\eta_0=0$, H_0 is then the free-particle Hamiltonian and the reflection coefficient of η_0 is $R_0(k)=0$ for the positive energies $E=k^2$. The reflection coefficient of H_1 is given by

$$R_1(k)=\frac{\gamma_1-ik}{\gamma_1+ik}R_0(k)$$

which is zero for $R_0(k)=0$. But it is the case of the reflectionless potential in (2.14) for $N=1$ and $b=(1/L\sqrt{2})$.

From SSQM let us suppose

$$V_-=v^2-v_x=1/2L^2. \tag{3.8}$$

Equation (3.8) is a very simple Riccati equation whose solution is given by substituting

$$v=-\psi_{0x}/\psi_0 \tag{3.9}$$

and we have

$$\psi_{0xx}/\psi_0=1/2L^2. \tag{3.10}$$

Here, ψ_0 is the solution of the zero-energy Schrödinger equation with

$$\left(-\frac{d^2}{dx^2}+v^2-v_x\right)\psi_0=0.$$

The form of the solution, ψ_0, of (3.10) is

$$\psi_0=\text{constant}\times\cosh\frac{x}{\sqrt{2}L} \tag{3.11}$$

and from (3.9) it follows that

$$v=-\frac{1}{\sqrt{2}L}\tanh\frac{x}{\sqrt{2}L}. \tag{3.12}$$

The superpartner to V_- has the form

$$V_+ = v^2 + v_x = \frac{1}{2L^2} - \frac{1}{L^2}\,\mathrm{sech}^2\,\frac{x}{L\sqrt{2}}. \tag{3.13}$$

Now, if we denote

$$\eta_0(x) = v^2 - v_x - 1/2L^2 = 0$$

$$\eta_1(x) = v^2 + v_x - \frac{1}{2L^2} = -\frac{1}{L^2}\,\mathrm{sech}^2\,\frac{x}{L\sqrt{2}}$$

we can see that H_1 is the superpartner to H_0.

Using results from SSQM we shall now demonstrate how to construct a symmetric reflectionless $\eta_j(x)$, $j = 1, 2, \ldots, N$. For arbitrary j we may now assume $\eta_{j-1}(x)$ to be known and define v_j by

$$\eta_{j+1} = v_j^2 - v_{jx} - E_j.$$

Then, the superpartner has the form

$$\eta_j = v_j^2 + v_{jx} - E_j.$$

The crucial point for the construction is that the supersymmetric reflectionless partner can be expressed via the eigenfunctions of the corresponding Hamiltonian. This can be seen from the equation ($j = 1$)

$$H_+ = A^+A^- = H_- + [A^+, A^-] = H_- + 2\frac{\mathrm{d}}{\mathrm{d}x}\,v = H_- - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\psi_0. \tag{3.14}$$

Hence

$$H_+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + E_1 + \eta_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\psi_0(E_1)$$

and

$$\eta_1 = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\psi_0(E_1) \tag{3.15}$$

for $\eta_0 = 0$.

We can apply this procedure to the z system (3.1). For higher $N = 2, 3, \ldots$ it is known (Calogero and Degasperis 1982, Sukumar 1986) that the symmetric reflectionless $\eta_N(x)$ may be expressed in terms of normalised bound-state eigenfunctions in the form

$$\eta_N(x) = -4\sum_{i=1}^{N}[\gamma_i\psi_N^2(E_i)]. \tag{3.16}$$

Using the results of SSQM a vector version of the NLS (VNLS) was presented (Hruby 1988) in the form

$$\mathrm{i}\partial_t\psi_N + \psi_{Nxx} - \eta_N\psi_N = 0 \tag{3.17a}$$

$$\left(\eta_N + 4\sum_{i=1}^{N}(\gamma_1|\psi_N(E_1)|^2)\right)_{xx} = 0 \tag{3.17b}$$

where $\psi_N(E_i)$ are bound states given by

$$\left(-\frac{d^2}{dx^2}+\eta_N(x)\right)\psi_N(E_i)=E_i\psi_N(E_i) \tag{3.18}$$

with η_N being 'symmetric' reflectionless potentials.

For the physical application, the interesting case is when

$$\eta_N(x)=-N(N+1)\frac{1}{2L^2}\operatorname{sech}^2\frac{x}{L\sqrt{2}}=-\frac{N(N+1)}{2}\eta_1(x). \tag{3.19}$$

Mathematically, (3.19) corresponds to the so-called Lame-Ains N-zones elliptical potential (Novikov 1980). In this case the VNLS system (3.17) has the form

$$\mathrm{i}\partial_t\psi_N+\psi_{Nxx}+\frac{N(N+1)}{2}|\psi_1|^2\psi_N=0 \tag{3.20a}$$

$$\left(\eta_N(x)+\frac{N(N+1)}{2}|\psi_1|^2\right)_{xx}=0. \tag{3.20b}$$

In these equations there exist the envelope solitary wave solutions

$$\psi_N=\exp(\mathrm{i}t/sL^2)\frac{1}{L}\operatorname{sech}\frac{\sqrt{N(N+1)}}{2L}x$$

for arbitrary $N=1,2,\ldots$.

To end this section we mention that these results are valid for all classes of so-called 'bona fide' potentials (Calogero and Degasperis 1982). An arbitrary couple of these potentials are given in the form

$$u^1(x)=\tfrac{1}{4}(\nu^2-2\nu_x) \qquad u^2(x)=\tfrac{1}{4}(\nu^2+2\nu_x)$$

where $\nu(x)$ is an arbitrary regular function; both $\nu(x)$ and its derivatives go to zero at infinity. Reflection (R) and transmission (T) coefficients corresponding to the potentials $u^{1,2}(x)$ are connected via the following relations:

$$R^1(k)=-R^2(k) \qquad T^1(k)=T^2(k). \tag{3.21}$$

It is interesting that supersymmetry appears as relation (3.19) between R and T coefficients of the two superpartner potentials $u^{1,2}$ (for $\nu=2v$, $u^{1,2}\equiv V_{+,-}$).

Reflectionless potentials which are discussed here are a special subclass of the 'bona fide' potentials and it is well known that these potentials are the multisoliton solutions of the KdV equation (Calogero and Degasperis 1982).

In the application to the z equations the time evolution of the reflectionless potentials η_N is different and is given by

$$\eta_{Ntt}-\eta_{Nxx}=4\left(\sum_{i=1}^{N}\gamma_i|\psi_N(E_i)|^2\right)_{xx}.$$

The numerical verification of the evolution of the soliton solutions given by (3.17a) and (3.17b) was given in Astrelin (1988). It is known that this admits another presentation first made by Makhankov and Myrzakulov (1986) by means of the factorisation method. The U(N) vector non-linear Schrödinger equation factorisation and the relation with the SSQM results (Sukumar 1986) was published by Hruby and Makhanov (1987).

4. The U(N) vector non-linear Schrödinger equation: factorisation and the relation with the SSQM results

The U(N) vector NLS has the form

$$\mathrm{i}\phi_{Nt} + \phi_{Nxx} + (\bar{\phi}_N \phi_N)\phi_N = 0 \tag{4.1}$$

where

$$\phi_N(x, t) = (\phi_{N,1}, \ldots, \phi_{N,m})^{\mathrm{T}} \qquad \bar{\phi}_N\phi_N = \sum_{j=1}^{m} |\phi_{N,j}|^2 \qquad m \geqslant N. \tag{4.2}$$

A new particular class of the soliton solutions of (4.1) has recently been obtained (Makhankov and Myrzakulov 1988) via the so-called factorisation method. We show that these solutions are equivalent to the reflectionless symmetric potentials of the one-dimensional Schrödinger equation and in the case when the potentials $\eta_N(x)$ have the form (2.14) it corresponds exactly to the results given in §§ 2 and 3 via SSQM. We can show this in the following way.

Write the solutions of (4.1) in the form

$$\phi_N(x, t) = C \exp(\mathrm{i} W)\phi_N(y) \tag{4.3}$$

where

$$\phi_N(y) = (\phi_{N,1}, \ldots, \phi_{N,m})^{\mathrm{T}} \qquad C = \mathrm{diag}(C_1, \ldots, C_m)$$

$$W = \mathrm{diag}(\theta_1, \ldots, \theta_m) \qquad \theta_j = \tfrac{1}{2}v(x - \tfrac{1}{2}vt) - \Lambda_j t \qquad y = x - vt$$

and put (4.3) into (4.1) to get

$$\phi''_{N,j} - \eta_N\phi_{N,j} = -\Lambda_j\phi_{N,j} \tag{4.4}$$

$$\eta_N = -\sum_{j=1}^{m} |C_j|^2 \phi^2_{N,j}.$$

Suppose the potential η_N to be in the form (2.14); then (4.4) becomes

$$\phi''_{N,j} + N(N+1)b^2 \operatorname{sech} by\phi_{N,j} = -\Lambda_j\phi_{N,j}. \tag{4.5}$$

It is known that (4.5) for arbitrary N has N eigenvalues $\Lambda_j = -j^2b^2, j = 1, 2, \ldots, N$. The corresponding eigenfunctions may be found by using the factorisation which is equivalent to the SSQM 'square root', mentioned above.

We can define $A_l^{\mp}$ in the same way as in (2.3), namely

$$A_l^{\mp} = \pm\frac{\mathrm{d}}{\mathrm{d}y} + lb \tanh by = \pm\frac{\mathrm{d}}{\mathrm{d}y} + v_l(y)$$

where v_l has the form (3.12) for $l = 1$ and $b = 1/L\sqrt{2}$.

Then in the same way as in SSQM we define

$$A^+_{l+1}\phi_{l,j} = \phi_{l+1,j} \tag{4.6}$$

$$A^-_{l+1}\phi_{l,j} = \phi_{l-1,j}. \tag{4.7}$$

From (4.6) and (4.7), using

$$\phi_{l,j} \equiv 0 \tag{4.8}$$

for $l > N$, we obtain all the solutions to (4.5). Some of them follow directly; e.g. for $N = j = l$ we get

$$A_N^+ \phi_{N,N} = 0$$

and from this

$$\phi_{N,N} \sim \operatorname{sech}^N by. \tag{4.9}$$

Generally, we have the recurrent formula

$$\phi_{l,j} = A_l^+ A_{l-1}^+ \ldots A_{j+1}^+ \phi_{j,j}. \tag{4.10}$$

Thus we obtain for $N = 1 = m$, i.e. $l = j = 1$, from (4.9)

$$\phi_{1,1} \sim \operatorname{sech} by \tag{4.11}$$

which corresponds to the SSQM relation (2.10) with ψ_0^- as in (3.11). In fact, the potential $V_+ = v_1^2 + v_{1x}$ has a zero-energy bound state whose eigenfunction is

$$\psi_0^+ \sim \operatorname{sech} \frac{x}{L\sqrt{2}}.$$

For $N = 2$ we have the two solutions corresponding to $\Lambda_1 = -b^2$ and $\Lambda_2 = -4b^2$. Then, from (4.9) it follows that

$$\phi_{2,2} \sim \operatorname{sech}^2 by \tag{4.12a}$$

and from (4.10) and (4.11) we get

$$\phi_{2,1} = A_2^+ \phi_{1,1} \sim \tanh by \operatorname{sech} by. \tag{4.12b}$$

So, we obtain from the relations (4.3) and (4.11) the known one-soliton solution of the U(1) NLS equation

$$\phi_1(x, t) = C \exp(\mathrm{i}\theta_1) \operatorname{sech} by \tag{4.13}$$

where $|C|^2 = 2b^2$.

For $N = m = 2$ it follows that the soliton solution to the U(2) VNLS is

$$\phi_2(x, 1) = \begin{pmatrix} c_1 \exp(\mathrm{i}\theta_1) \sinh by \\ C_2 \exp(\mathrm{i}\theta_2) \end{pmatrix} \operatorname{sech}^2 by \tag{4.14}$$

where $|C_1|^2 = |C_2|^2 = 6b^2$. Analogous results hold for $N = m = 3$ and so on.

The general expression for the symmetric reflectionless potentials $\eta_N(x, t)$ in (4.4) can be given following Sukumar (1986)

$$\eta_N = -2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \det D_N \tag{4.15}$$

where the elements of the matrix D_N are given by

$$[D_N]_{Jk} = \tfrac{1}{2}(\gamma_k)^{J-1}[\exp(\gamma_k x) + (-1)^{J+k} \exp(-\gamma_k x)] \tag{4.16}$$

and the normalised eigenfunctions for the eigenenergy $E_j = -\gamma_j^2 \equiv \Lambda_j$ may be written in the form

$$\tilde{\phi}_N(E_j) = \left(\frac{\gamma_j}{2} \sum_{k \neq j}^{N} |\gamma_k^2 - \gamma_j^2|\right)^{1/2} [D_N^{-1}]_{jN} \tag{4.17}$$

where $j = 1, 2, \ldots, N$.

For $N=2$ from the relations (4.15)-(4.17) it follows that

$$D_2=\begin{pmatrix} \cosh \gamma_1 x & \sinh \gamma_2 x \\ \gamma_1 \sinh \gamma_1' x & \gamma_2 \cosh \gamma_2 x \end{pmatrix} \tag{4.18}$$

$$\eta_2(x)=-2(\gamma_2^2-\gamma_1^2)\frac{\gamma_2^2 \cosh \gamma_1 x+\gamma_1^2 \sinh \gamma_2 x}{(\gamma_2 \cos \gamma_2 x \cosh \gamma_1 x-\gamma_1 \sinh \gamma_2 x \sinh \gamma_1 x)^2} \tag{4.19}$$

$$\begin{aligned}\tilde{\phi}_2(E_1)&=\left(\frac{\gamma_1}{2}(\gamma_2^2-\gamma_1^2)\right)^{1/2}\frac{\cosh \gamma_2 x}{\det D_2}\\ &=(\varkappa-\nu)\frac{\cosh \nu y-\cosh \varkappa y-\sinh \nu y-\sinh \varkappa y}{\varkappa \cosh \nu y+\nu \cosh \varkappa y}\end{aligned} \tag{4.20a}$$

$$\begin{aligned}\tilde{\phi}_2(E_2)&=\left(\frac{\gamma_2}{2}(\gamma_2^2-\gamma_1^2)\right)^{1/2}\frac{\cosh \gamma_1 x}{\det D_2}\\ &=(\varkappa+\nu)\frac{\sinh \nu y+\sinh \varkappa y-\cosh \nu y-\cosh \varkappa y}{\varkappa \cosh \nu y+\nu \cosh \varkappa y}\end{aligned} \tag{4.20b}$$

where $\varkappa=-(\gamma_1-\gamma_2)$ and $\nu=\gamma_1+\gamma_2$.

Equations (4.19) and (4.20) coincide with (22) and (25), (26) of Makhankov and Myrzakulov (1986).

We can also see that

$$\eta_2(x)=-4(\gamma_2\tilde{\phi}_2^2(E_2)+\gamma_1\tilde{\phi}_2^2(E_1)] \tag{4.21}$$

and in particular, if $\gamma_2^2=4\gamma_1^2$ the resulting potential is

$$\eta_2(x)=-6\gamma_1^2 \operatorname{sech}^2 \gamma_1 x \tag{4.22}$$

i.e. $\eta_2(x)$ exactly corresponds to the potential (2.14) for $N=2$.

There exist two bound states at $\Lambda_1=-\gamma_1^2=-b^2$, $\Lambda_2=-4\gamma_1^2=-4b^2$ and (4.20*a*, *b*) correspond to (4.14).

These new soliton solutions of the VNLS are equivalent to the known solutions in SSQM (Sukumar 1986).

5. Conclusions

In this paper, after introduction to SSQM, we have shown the general connection between N-soliton solutions of KdV, SSQM, the inverse scattering method and the construction of the reflectionless potentials.

The new features of the paper seem to be the following observations.

(i) The Miura transformation in some sense represents the supersymmetric 'square root'.

(ii) The symmetry between reflection and transmition coefficients for 'bona fide' potentials in quantum mechanics is connected with supersymmetry.

(iii) The application of SSQM to the VNLS equation provides the possibility of investigating the soliton sector of certain non-integrable systems such as the z system. The symmetric reflectionless potentials are obtained here as linear combinations of the eigenvalue solutions.

It should be noted that symmetric reflectionless SSQM potentials and those obtained via the familiar factorisation method naturally coincide up to reparametrisation.

The new result (iii) has possible applications to plasma physics and non-linear optics.

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